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The index of maximum ambiguous density for irreducible non-powerful sign pattern matrices ☆

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ABSTRACT

Let $INS_{n,p}$ be the set of $n \times n$ irreducible non-powerful (generalized) sign pattern matrices with period p, and let $A \in INS_{n,p}$. In this paper, we introduce a new parameter called the index of maximum ambiguous density of A. Furthermore, the generalized index of maximum ambiguous density of A, which generalizes the concept of the index of maximum ambiguous density, is introduced. Moreover, some bounds on these indices are obtained, and we exhibit a system of gaps in the set of the index of maximum ambiguous density for $A \in INS_{n,p}$. Finally, the index and the generalized index of maximum ambiguous density for irreducible non-powerful zero-symmetric sign pattern matrices are discussed.

1. Introduction

The sign of a real number a, denoted by $sgn(a)$, is defined to be 1, -1 or 0, according to $a > 0, a < 0$, or $a = 0$. The sign pattern of a real matrix A, denoted by $sgn(A)$, is the $(0, 1, -1)$-matrix obtained from A by replacing each entry by its sign.

For a square sign pattern matrix A, notice that in the computations of the entries of the powers $A^k (k = 1, 2, \ldots)$, the ambiguous sign may arise when a positive sign is added to a negative sign. So a new symbol # has been introduced to denote the ambiguous sign [3], and we call a matrix with entries from the set $\Gamma = \{0, 1, -1, \#\}$ a generalized sign pattern matrix.

The addition and multiplication involving the symbol # are defined as follows (the addition and multiplication which do not involve # are obvious):

$$(-1) + 1 = 1 + (-1) = \#; \qquad a + \# = \# + a = \# \quad (\text{for all} \quad a \in \Gamma);$$
$$0 \cdot \# = \# \cdot 0 = 0; \qquad b \cdot \# = \# \cdot b = \# \quad (\text{for all} \quad b \in \Gamma \setminus \{0\}).$$

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From now on we assume that all the matrix operations considered in this paper are operations of the matrices over the set Γ.

Definition 1.1 (*[3]*). Let A be a generalized sign pattern matrix of order n and $A, A^2, A^3, \cdots$ be the sequence of powers of A. Suppose A^l is the first power that is repeated in the sequence. Namely, suppose l is the least positive integer such that there is a smallest positive integer p such that $A^l = A^{l+p}$. Then l is called the base of A denoted by $l(A)$, and p is called the period of A denoted by $p(A)$.

A sign pattern matrix A is reducible if there is a permutation pattern P such that $P^{\mathrm{T}}AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where B and D are non-empty and square. A sign pattern matrix A is said to be irreducible if it is not reducible [5].

A square generalized sign pattern matrix A is called powerful if each power of A contains no # entry. A is called non-powerful if A is not powerful [3].

A non-negative square matrix A is primitive if some power $A^k > 0$. For a generalized sign pattern matrix A, we use $|A|$ to denote the (0, 1)-matrix obtained from A by replacing each non-zero entry by 1. For convenience, a square generalized sign pattern matrix A is called primitive if $|A|$ is primitive. A square irreducible sign pattern matrix A is called imprimitive if it is not primitive [9].

Let $INS_{n,\,p}$ denote the set of $n \times n$ irreducible non-powerful (generalized) sign pattern matrices with period p.

Definition 1.2 (*[4]*). Let $A \in INS_{n,\,1}$ (namely, A is primitive) and let k be an integer with $1 \le k \le n$. The kth local base $l_A(k)$ is the smallest power of A for which there exist k rows each of whose entries is ambiguous sign (i.e., #).

The index and the generalized index of maximum density for irreducible Boolean matrices are investigated in many years (see [2,7]). Recently, Shao and You [9] studied the bases of irreducible non-powerful sign pattern matrices.

Let $A \in INS_{n,\,p}$. Inspired by the concept of the index of maximum density for irreducible Boolean matrices, we introduce a new parameter called the index of maximum ambiguous density of A in Section 3. Furthermore, we generalize the concept of the index of maximum ambiguous density to a related parameter called the generalized maximum ambiguous density index of A in Section 4. At last, the index and the generalized index of maximum ambiguous density for irreducible non-powerful zero-symmetric sign pattern matrices are discussed.

2. Preliminaries

In this section, we introduce some definitions, notations and lemmas which are needed for obtaining our main results.

It is known that if a sign pattern matrix A is irreducible with index of imprimitivity p (i.e., $p = p(|A|)$), then A is permutation similar to a sign pattern matrix of the following block partitioned form (called the "imprimitive normal form" of A):

$$QAQ^{\mathrm{T}} = \begin{pmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{p-1} \\ A_p & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where the zero blocks along the diagonal are square, and Q is a permutation sign pattern matrix. If the block A_i is of size $n_i \times n_{i+1}$ ($i = 1, 2, \ldots, p$, where the subscripts are read mod p), then we denote the "imprimitive normal form" of A as $(n_1, A_1, n_2, A_2, \ldots, n_p, A_p, n_1)$, or simply $(A_1, \ldots, A_p)$ in case the sizes of the blocks need not be indicated explicitly (see [9] or [8]).

For convenience, define $A_{j+p} = A_j$ for all j and define $A_i(m) = A_iA_{i+1}\cdots A_{i+m-1}$ to be the product of m successive sign pattern matrices. Therefore, each block A_i contains no zero row and no zero column $(1 \le i \le p)$.

Let m be a non-negative integer and Z_i be a sign pattern matrix of the size $n_i \times n_{i+1}(i = 1, 2, \dots, p)$. Define $(Z_1, Z_2, \dots, Z_p)_m$ to be the block partitioned sign pattern matrix $(A_{ij})(i,\ j = 1, 2, \dots, p)$ with the blocks

$$A_{ij} = \begin{cases} Z_i, & \text{if } j - i \equiv m (\text{mod } p), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see from this definition that $(A_1, \dots, A_p)_1 = (A_1, \dots, A_p)$.

Using these notations together with the recursive computations, we have the following formula for the power A^m of $A = (A_1, \dots, A_p)$:

$$A^m = (A_1(m), A_2(m), \dots, A_p(m))_m.$$

We shall make use of the following lemmas in this paper.

Lemma 2.1 ([9]). *Let X and Y be $m \times n$ and $n \times m$ generalized sign pattern matrices without zero rows or zero columns. Then $|l(XY) - l(YX)| \le 1$.*

Lemma 2.2 ([9]). *Let $A = (A_1, \dots, A_p)$ be an irreducible sign pattern matrix of the "imprimitive normal form" with index of imprimitivity p (i.e., $p(|A|) = p$). Then the following conditions are equivalent:*

(1) *A is not powerful.*
(2) *There exists some i such that $A_i(p)$ is not powerful.*
(3) *For each $j = 1, 2, \dots, p$, $A_j(p)$ is not powerful.*

A generalized sign pattern matrix with all entries equal to # is denoted by $\#J$.

Lemma 2.3 ([9]). *Let $A = (A_1, \dots, A_p) \in INS_{n,\ p}$. Then*

(1) *There exists some positive integer k such that $A_i(k) = \#J$ for all $i = 1, 2, \dots, p$.*
(2) *If $A_i(k) = \#J$ for all $i = 1, 2, \dots, p$, then $A_i(k+1) = \#J$ for all $i = 1, 2, \dots, p$.*
(3) *$p(A) = p$ and $l(A) = \min\{k \mid A_i(k) = \#J$ for all $i = 1, 2, \dots, p\}$.*

Remark 1. By Lemma 2.3, if $A \in INS_{n,\ 1}$, then the base $l(A) = \min\{k \mid A^k = \#J\}$.

In addition, each $A_i(p)(1 \le i \le p)$ is primitive and non-powerful.

Lemma 2.4 ([9]). *Let $A = (n_1, A_1, n_2, A_2, \dots, n_p, A_p, n_1) \in INS_{n,\ p}$. Suppose $1 \le i_1 < \cdots < i_t \le p$ and $l_{i_j} = l(A_{i_j}(p))$ is the base of $A_{i_j}(p)(1 \le j \le t)$. Then*

$$l(A) \le p \cdot \max\{l_{i_1},\ l_{i_2},\ \cdots,\ l_{i_t}\} + p - t.$$

Corollary 2.5 ([9]). *Let $A = (n_1, A_1, n_2, A_2, \dots, n_p, A_p, n_1) \in INS_{n,\ p}$ and let $m = \min\{n_1, n_2, \dots, n_p\}$. Then $l(A) \le p[2(m-1)^2 + m + 1] - 1$.*

Lemma 2.6 ([9]). *Let $A \in INS_{n,\ p}$ and let $n = pr + s$, where $r = \lfloor \frac{n}{p} \rfloor$ and $0 \le s \le p - 1$($\lfloor x \rfloor$ is the largest integer not exceeding x). Then*

$$l(A) \le p[2(r-1)^2 + r] + s.$$

Let D_1 be the digraph with the set $V = \{i \mid 1 \le i \le n\}$ of vertices and the set $E = \{(i,\ i+1) \mid 1 \le i \le n-1\} \cup \{(n,\ 1),\ (n-1,\ 1)\}$ of arcs.

Lemma 2.7 ([9]). *Let $A \in INS_{n,\ 1}$ with $D(A)$ as its associated digraph.*

(1) *$l(A) \le 2n^2 - 3n + 2$. Moreover, the equality holds if and only if $D(A)$ is isomorphic to D_1.*
(2) *For each integer k with $2n^2 - 4n + 5 < k < 2n^2 - 3n + 1$, there is no sign pattern matrix $A \in INS_{n,\ 1}$ such that $l(A) = k$.*

Lemma 2.8 ([4]). *Let $A \in INS_{n,\ 1}$ with $D(A)$ as its associated digraph. Then $l_A(k) \le 2n^2 - 4n + 2 + k$ for $1 \le k \le n$. Moreover, the equality holds if and only if $D(A)$ is isomorphic to D_1.*

3. The index of maximum ambiguous density

Now let us give the definition of the index of maximum ambiguous density.

Definition 3.1. Let $A \in INS_{n,\,p}$. The index of maximum ambiguous density of A is defined as the least integer $\phi = \phi(A)$ such that the number of ambiguous signs (i.e., #) in A^{ϕ} is maximized in all powers of A.

Let $A \in INS_{n,\,p}$. Denote maximum ambiguous density of A by $\beta(A) = \max_{m \in Z^+} \|A^m\|_{\#}$, where $\|M\|_{\#}$ denotes the number of # in a generalized sign pattern matrix M. It follows that $\phi(A) = \min\{k \in Z^+ \mid \|A^k\|_{\#} = \beta(A)\}$.

Let $\Phi_{n,\,p}$ be the set of the indices of maximum ambiguous density $\phi(A)$ for $A \in INS_{n,\,p}$. Let $\phi(n,\,p) = \max\{\phi(A) \mid A \in INS_{n,\,p}\}$.

Remark 2. Let $A = (A_1, \ldots, A_p) \in INS_{n,\,p}$. It follows from Lemma 2.3 that the index of maximum ambiguous density of A (i.e., $\phi(A)$) is well defined and finite.

It is easy to see that $\beta(A) \le n^2$ and $\phi(A) \le l(A) + p - 1$.

We begin with the discussion on the maximum ambiguous density $\beta(A)$ and the index of maximum ambiguous density $\phi(A)$ for $A \in INS_{n,\,1}$.

Lemma 3.2. *If $A \in INS_{n,\,1}$ (namely, A is primitive), then*

$$\beta(A) = n^2 \quad \text{and} \quad \phi(A) = l(A) = \min\{k \mid A^k = \#J\}.$$

Proof. Since $A \in INS_{n,\,1}$, it follows from Remark 1 that

$$p(A) = p = 1 \quad \text{and} \quad l(A) = \min\{k \mid A^k = \#J\}.$$

Moreover, by the definition of the index of maximum ambiguous density, it follows that $\beta(A) = n^2$ and $\phi(A) = l(A) = \min\{k \mid A^k = \#J\}$. $\square$

Combining Lemmas 2.7 and 3.2, we have the following corollary.

Corollary 3.3. $\phi(n,\,1) = 2n^2 - 3n + 2$.

Therefore, the index of maximum ambiguous density need to be studied for irreducible imprimitive non-powerful sign pattern matrices mainly.

Lemma 3.4. *Let $A = (n_1, A_1, n_2, \ldots, n_p, A_p, n_1) \in INS_{n,\,p}$. Denote*

$$\#B_1 = \begin{pmatrix} 0 & \#J & 0 & \cdots & 0 \\ 0 & 0 & \#J & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \#J \\ \#J & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \#B_2 = (\#B_1)^2 = \begin{pmatrix} 0 & 0 & \#J & 0 & \cdots & 0 \\ 0 & 0 & 0 & \#J & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \#J \\ \#J & 0 & 0 & 0 & \cdots & 0 \\ 0 & \#J & 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots,$$

$$\#B_{p-1} = (\#B_1)^{p-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \#J \\ \#J & 0 & \cdots & 0 & 0 \\ 0 & \#J & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \#J & 0 \end{pmatrix}, \quad \textit{and} \quad \#B_0 = \begin{pmatrix} \#J & 0 & \cdots & 0 \\ 0 & \#J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \#J \end{pmatrix},$$

where each $\#B_j (j = 0, 1, \ldots, p-1)$ is an $n \times n$ generalized sign pattern matrix partitioned in the same block form as A. Then the base of A is

$$l(A) = \min\{m \in Z^+ \mid A^m = \#B_j,\ \text{where } m \equiv j (\mathrm{mod}\ p), 0 \le j \le p-1\}.$$

Proof. If $m = tp + j(0 \le j \le p-1)$, $A^m = \#B_j$, then $A^{m+p} = \#B_j \cdot A^p = \#B_j$.
Hence $A^m = A^{m+p}$, and then $l(A) \le m$ by Definition 1.1.
Conversely, let $l = l(A)$, and $l \equiv j (\bmod\ p)(0 \le j \le p-1)$.
For $t \ge \max_{1\le j\le p}\{l(A_j(p))\}$, we have $A^{tp} = \#B_0$. It follows that $A^l = A^{l+tp} = A^l \cdot \#B_0 = A^j \cdot \#B_0 = \#B_j$. Thus the equality is established. □

Lemma 3.5. *Let $A = (A_1, \ldots, A_p) \in INS_{n,\,p}$. Then*

$$|l(A_i(p)) - l(A_j(p))| \le 1, \quad \text{where } i,\ j = 1, 2, \ldots, p.$$

Proof. If $i = j$, then $|l(A_i(p)) - l(A_j(p))| = 0 < 1$. Now suppose $i < j$.
Let $X = A_i(j-i)$ and $Y = A_j(p-j+i)$. Thus $A_i(p) = XY$ and $A_j(p) = YX$.
Combining Lemmas 2.1–2.3, $|l(XY) - l(YX)| \le 1$, the result is obtained. □

Lemma 3.6. *Let $A = (A_1, \ldots, A_p) \in INS_{n,\,p}$ and let $l_i = l(A_i(p))$. Then*

$$p(l_i - 1) < l(A) \le p(l_i + 1) \quad \text{for all} \quad i = 1, 2, \ldots, p.$$

Proof. By Lemma 3.5, we have $l_i + 1 \ge \max_{1\le j\le p} l_j$ for all $i = 1, \ldots, p$. Thus $A^{p(l_i+1)} = \#B_0$. It follows that $l(A) \le p(l_i + 1)$. On the other hand, $(A_i(p))^{l_i-1} \lneqq \#J$, we have $A^{p(l_i-1)} \lneqq \#B_0$, and then $l(A) > p(l_i - 1)$. □

Let $C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$ be a $p \times p$ circulant matrix. The cyclic period of a row vector $(n_1, n_2, \ldots, n_p)$, denoted by $\tau(n_1, n_2, \ldots, n_p)$, is defined as the least positive integer j such that $C^j(n_1, \ldots, n_p)^{\mathrm{T}} = (n_1, \ldots, n_p)^{\mathrm{T}}$. Clearly, $\tau(n_1, \ldots, n_p) \mid p$.

Theorem 3.7. *Suppose $A = (n_1, A_1, n_2, A_2, \ldots, n_p, A_p, n_1) \in INS_{n,\,p}$. Then*

$$\beta(A) = \max_{m\in Z^+} \|A^m\|_\# = \sum_{i=1}^{p} n_i^2,$$

$$\phi(A) = \min\{m \in Z^+ \mid m \ge l(A) \quad \text{and} \quad \tau \mid m\} = \tau \cdot \left\lceil \frac{l(A)}{\tau} \right\rceil,$$

where $\tau = \tau(n_1, n_2, \ldots, n_p)$, and $\lceil x \rceil$ denotes the least integer not less than x.

Proof. Assume $m \equiv j (\bmod\ p)(0 \le j \le p-1)$. By Lemma 3.4, if $m \ge l(A)$, then $A^m = \#B_j$, and $\|A^m\|_\# = \sum_{i=1}^p n_i n_{i+j}$; if $m < l(A)$, then $A^m \lneqq \#B_j$, and $\|A^m\|_\# < \sum_{i=1}^p n_i n_{i+j}$.
Since $\sum_{i=1}^p n_i^2 - \sum_{i=1}^p n_i n_{i+j} = \frac{1}{2}\sum_{i=1}^p (n_i - n_{i+j})^2 \ge 0$, then $\|A^m\|_\# \le \sum_{i=1}^p n_i^2$, where the equality holds if and only if $n_i = n_{i+j}$ for all $i = 1, 2, \ldots, p$, that is, $C^j(n_1, n_2, \ldots, n_p)^{\mathrm{T}} = (n_1, n_2, \ldots, n_p)^{\mathrm{T}}$, and then $\tau(n_1, n_2, \ldots, n_p) \mid j$. In particular, the above equality holds when $j = p$, and then $\beta(A) = \max_m \|A^m\|_\# = \sum_{i=1}^p n_i^2$.
Furthermore, note that $\tau(n_1, n_2, \ldots, n_p) \mid p$, $m \equiv j (\bmod\ p)$, and

$$\begin{aligned} \|A^m\|_\# = \sum_{i=1}^p n_i^2 &\Leftrightarrow m \ge l(A) \quad \text{and} \quad \sum_{i=1}^p n_i^2 = \sum_{i=1}^p n_i n_{i+j} \\ &\Leftrightarrow m \ge l(A) \quad \text{and} \quad \tau(n_1, n_2, \ldots, n_p) \mid j \\ &\Leftrightarrow m \ge l(A) \quad \text{and} \quad \tau(n_1, n_2, \ldots, n_p) \mid m. \end{aligned}$$

Thus $\phi(A) = \min\{m \in Z^+ \mid m \ge l(A) \text{ and } \tau \mid m\} = \tau \cdot \left\lceil \frac{l(A)}{\tau} \right\rceil$. □

Corollary 3.8. *Suppose* $A = (n_1, A_1, n_2, A_2, \ldots, n_p, A_p, n_1) \in INS_{n,\ p}$ *and let* $m = \min\{n_1, n_2, \ldots, n_p\}$*. Then* $\phi(A) \le p \cdot [2(m-1)^2 + m + 1]$.

Proof. By Corollary 2.5, we have $l(A) \le p[2(m-1)^2 + m + 1] - 1$. It follows from Theorem 3.7 that

$$\begin{aligned}\phi(A) &= \tau \cdot \left\lceil \frac{l(A)}{\tau} \right\rceil \le p \cdot \left\lceil \frac{l(A)}{p} \right\rceil \le p \cdot \left\lceil \frac{p \cdot [2(m-1)^2 + m + 1] - 1}{p} \right\rceil \\ &= p \cdot [2(m-1)^2 + m + 1]. \quad \square\end{aligned}$$

Theorem 3.9. *Let* $n = pr + s$*, where* $r = \lfloor \frac{n}{p} \rfloor$ *and* $0 \le s \le p-1$*. Then*

$$\phi(n,\ p) \le \begin{cases} p(2r^2 - 3r + 2) & (s = 0), \\ p(2r^2 - 3r + 3) & (1 \le s \le p-1). \end{cases}$$

Proof. Let $A \in INS_{n,\ p}$. By Theorem 3.7, it follows that

$$\phi(A) = \tau \cdot \left\lceil \frac{l(A)}{\tau} \right\rceil \le p \cdot \left\lceil \frac{l(A)}{p} \right\rceil, \quad \text{where } \tau = \tau(n_1, n_2, \ldots, n_p).$$

By Lemma 2.6, we have $l(A) \le p[2(r-1)^2 + r] + s$. Hence

$$\phi(A) \le p \cdot \left\lceil \frac{l(A)}{p} \right\rceil \le \begin{cases} p(2r^2 - 3r + 2) & (s = 0), \\ p(2r^2 - 3r + 3) & (1 \le s \le p-1). \end{cases} \quad \square$$

Secondly, we exhibit a system of "gaps" in $\Phi_{n,\ p}$.

Lemma 3.10. *For each integer* $n \ge 1$, $\Phi_{n,\ 1} \subseteq \Phi_{n+1,\ 1}$.

Proof. Let $A = (a_{ij}) \in INS_{n,\ 1}$ such that $\phi(A) = l(A) = t$. Then $A^t = \#J$.

Construct a primitive non-powerful sign pattern matrix B of order $n+1$ such that $B = \begin{pmatrix} A & B_1 \\ B_2 & a_{n+1,\ n+1} \end{pmatrix}$, where B_1 is the same as the last column of A, and B_2 is the same as the last row of A, and $a_{n+1,\ n+1} = a_{n,\ n}$.

It is easy to show that $B^t = \#J$, and then $\phi(B) = l(B) = t$. Therefore, for each integer $n \ge 1$, $\Phi_{n,\ 1} \subseteq \Phi_{n+1,\ 1}$. $\square$

The following theorem establishes a relation between the gaps in $\Phi_{n,\ 1}$ and the gaps in $\Phi_{n,\ p}$, and hence exhibits a system of gaps in $\Phi_{n,\ p}$.

Theorem 3.11. *Suppose* $n = pr + s$*, where* $r = \lfloor \frac{n}{p} \rfloor$ *and* $0 \le s \le p-1$*. If* $k \notin \Phi_{r,\ 1}$ *for all* $k_1 \le k \le k_2$*, then* $m \notin \Phi_{n,\ p}$ *for all* $k_1 p < m \le k_2 p$.

Proof. Let $k \notin \Phi_{r,\ 1}$ for all $k_1 \le k \le k_2$, and $m \in \Phi_{n,\ p}$, where $k_1 p < m \le k_2 p$. Then $m = \phi(A)$ for some $A \in INS_{n,\ p}$. Without loss of generality, we may assume that A is in the imprimitive normal form $A = (n_1, A_1, n_2, \ldots, n_p, A_p, n_1)$.

Since $n_1 + \cdots + n_p = n = pr + s < p(r+1)$, there exists some $n_j \le r$.

Notice that $A_j(p) = A_j A_{j+1} \cdots A_{j+p-1}$ is primitive non-powerful of size $n_j \times n_j$. Since $n_j \le r$, by Lemma 3.10, $l_j \triangleq l(A_j(p)) = \phi(A_j(p)) \in \Phi_{n_j,\ 1} \subseteq \Phi_{r,\ 1}$.

On the other hand, by Lemma 3.6, $p(l_j - 1) < m \le p(l_j + 1)$. Therefore, $p(l_j - 1) < k_2 p$ and then $l_j \le k_2$, $k_1 p < p(l_j + 1)$ and then $k_1 \le l_j$.

Thus $k_1 \le l_j \le k_2$ and $l_j \in \Phi_{r,\ 1}$, which is a contradiction. $\square$

Corollary 3.12. *Suppose* $n = pr + s$*, where* $r = \lfloor \frac{n}{p} \rfloor \ge 7$ *and* $0 \le s \le p-1$*. Then* $m \notin \Phi_{n,\ p}$ *for all* $2p(r^2 - 2r + 3) < m \le pr(2r-3)$.

Proof. By Lemma 2.7, $k \notin \Phi_{r,\ 1}$ for all $2r^2 - 4r + 5 < k < 2r^2 - 3r + 1$. Denote $k_1 = 2r^2 - 4r + 6$ and $k_2 = 2r^2 - 3r$. It follows from Theorem 3.11 that $m \notin \Phi_{n,\ p}$ for all $k_1 p < m \le k_2 p$. This completes the proof. □

Remark 3. This corollary exhibits a system of "gaps" in $\Phi_{n,\ p}$.

4. The generalized index of maximum ambiguous density

In this section, we introduce the generalized index of maximum ambiguous density, which are generalizations of the index of maximum ambiguous density.

Let $A \in INS_{n,\ p}$ and $V(A) = \{1, 2, \dots, n\}$. Let $\beta_A^j(i)$ denote the number of ambiguous signs (i.e., #) in the ith row of A^j. If $X \subseteq \{1, 2, \dots, n\} = V(A)$, we define $\beta_A^j(X) := \sum_{i \in X} \beta_A^j(i)$ and $\beta_A(X) := \max_{j \in Z^+}\{\beta_A^j(X)\}$.

Definition 4.1. The k-generalized maximum ambiguous density of A :

$$\beta_A(k) := \max_{|X|=k}\{\beta_A(X)\}\ (1 \le k \le n).$$

The k-generalized index of maximum ambiguous density of A:

$$\phi_A(k) := \min\{m \in Z^+ \mid \textit{there exists } X \subseteq V(A)\ (|X| = k)\ \textit{such that } \beta_A^m(X) = \beta_A(k)\}.$$

Moreover, we define $\phi(n,\ p,\ k) := \max\{\phi_A(k) \mid A \in INS_{n,\ p}\}\ (1 \le k \le n)$.

Remark 4. Obviously, $\beta_A(n) = \beta(A)$, $\phi_A(n) = \phi(A)$ and $\phi(n,\ p,\ n) = \phi(n,\ p)$.

In addition, it follows from Lemma 2.3 that the k-generalized index of maximum ambiguous density of A (i.e., $\phi_A(k)$) is well defined and finite.

It is easy to see that $\beta_A(k) \le nk$ and $\phi_A(k) \le \phi_A(n) = \phi(A) \le l(A) + p - 1$.

Firstly, $\beta_A(k)$ and $\phi_A(k)$ are studied for $A \in INS_{n,\ 1}$ (namely, A is primitive).

Lemma 4.2. *Let k be an integer with $1 \le k \le n$. If $A \in INS_{n,\ 1}$, then*

$$\beta_A(k) = kn \quad \textit{and} \quad \phi_A(k) = l_A(k).$$

Proof. Since $A \in INS_{n,\ 1}$, it follows from Definition 1.2 that the kth local base $l_A(k)$ is the smallest power of A for which there exist k rows each of whose entries is ambiguous sign (i.e., #).

Moreover, by the definition of k-generalized index of maximum ambiguous density, we have $\beta_A(k) = kn$ and $\phi_A(k) = l_A(k)$ immediately. □

Combining Lemmas 2.8 and 4.2, we have

Corollary 4.3. *Let k be an integer with $1 \le k \le n$. Then*

$$\phi(n,\ 1,\ k) = 2n^2 - 4n + 2 + k.$$

Note that the signed digraph $S(A)$ for $A \in INS_{n,\ p}$ is a p-partite digraph with the partition $(V_1, V_2, \dots, V_p)$ and $|V_i| = n_i$. The k-generalized maximum ambiguous density of A can be characterized as follows.

Theorem 4.4. *Let $A = (n_1, A_1, n_2, A_2, \dots, n_p, A_p, n_1) \in INS_{n,\ p}$. Denote the set of the multiset $\{n_1, n_2, \dots, n_p\}$ (since some n_i may be equal) by $\{\eta_1, \eta_2, \dots, \eta_m\}$ with $\eta_1 > \eta_2 > \dots > \eta_m$. Suppose* there are *$x_i$ subsets which are $\eta_i (i = 1, 2, \dots, m)$ in the partition of $S(A)$, where $x_1 + \dots + x_m = p$.*

Then

$$\beta_A(k) = \begin{cases} k\eta_1 & (1 \le k \le x_1\eta_1), \\ \displaystyle\sum_{j=1}^{i} x_j\eta_j^2 + \left(k - \sum_{j=1}^{i} x_j\eta_j\right) \cdot \eta_{i+1} & \left(1 + \displaystyle\sum_{j=1}^{i} x_j\eta_j \le k \le \sum_{j=1}^{i+1} x_j\eta_j\right), \\ \displaystyle\sum_{j=1}^{m-1} x_j\eta_j^2 + \left(k - \sum_{j=1}^{m-1} x_j\eta_j\right) \cdot \eta_m & \left(1 + \displaystyle\sum_{j=1}^{m-1} x_j\eta_j \le k \le n\right), \end{cases}$$

where $1 \le i \le m-2$. *In particular,* $\beta_A(n) = \sum_{j=1}^{m} x_j\eta_j^2 = \sum_{j=1}^{p} \eta_j^2 = \beta(A)$.

Proof. Note that $\eta_1 > \eta_2 > \cdots > \eta_m$ and there are x_i subsets which are $\eta_i (i = 1, 2, \ldots, m)$ in the partition of $S(A)$. By the definition of k-generalized maximum ambiguous density, similarly as the proof of Theorem 3.7, it is not difficult to obtain the desired results. □

Finally, the k-generalized index of maximum ambiguous density $\phi(n, p, k)$ with $p > 1$ is investigated. (The case of $p = 1$ is settled in Corollary 4.3.)

Lemma 4.5. *Let* $A = (A_1, \ldots, A_p) \in INS_{n,\,p}$. *Let* $l_i = l(A_i(p))$ *be the base of* $A_i(p)$. *Then for every* $1 \le k \le n$, $\phi_A(k) \le p \cdot (\max_{1\le i\le p}\{l_i\})$.

Proof. Let $l = \max_{1\le i\le p}\{l_i\}$. Since $A = (A_1, \ldots, A_p) \in INS_{n,\,p}$, we have

$$A^{pl} = \begin{pmatrix} [A_1(p)]^l & 0 & \cdots & 0 \\ 0 & [A_2(p)]^l & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [A_p(p)]^l \end{pmatrix} = \begin{pmatrix} \#J & 0 & \cdots & 0 \\ 0 & \#J & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \#J \end{pmatrix}.$$

By Theorem 4.4, it follows that $\phi_A(k) \le p \cdot l \le p \cdot (\max_{1\le i\le p}\{l_i\})$. □

Lemma 4.6. *Let* $A = (n_1, A_1, n_2, A_2, \ldots, n_p, A_p, n_1) \in INS_{n,\,p}$. *Suppose there exist some integers* i, j *such that* $n_i = r$ *and* $n_j = \max_{1\le m\le p}\{n_m\}$, *where* $1 \le i, j \le p$. *Let* $t \equiv j - i (\mathrm{mod}\ p)$. *Let* $l_i(k) = l_{A_i(p)}(k)$ *be the kth local base of* $A_i(p)$. *Then*

$$\phi_A(k) \le p \cdot l_i(k) + t \le p \cdot (2r^2 - 4r + 2 + k) + t \quad \text{for } 1 \le k \le r.$$

Proof. If there exists some integer i with $1 \le i \le p$ such that $n_i = r$, by Remark 1, $A_i(p)$ is primitive non-powerful of order r. Note that

$$A^{p\cdot l_i(k)} = \begin{pmatrix} [A_1(p)]^{l_i(k)} & 0 & \cdots & 0 \\ 0 & [A_2(p)]^{l_i(k)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [A_p(p)]^{l_i(k)} \end{pmatrix}.$$

Consequently, there are k rows $\{i_1, \ldots, i_k\}$ in $[A_i(p)]^{l_i(k)}$ such that each of whose entries is ambiguous sign (i.e., #). Let $X = \{i_1, \ldots, i_k\}$ with $|X| = k (1 \le k \le r)$.

Considering $A^{p\cdot l_i(k)+t}$, where $t \equiv j - i (\mathrm{mod}\ p)$ and $n_j = \max_{1\le m\le p}\{n_m\}$, we have $\beta_A^{p\cdot l_i(k)+t}(X) = kn_j$. By Theorem 4.4, $\beta_A^{p\cdot l_i(k)+t}(X) = \beta_A(k)$.

It follows from Lemma 2.8 that $l_i(k) \le 2r^2 - 4r + 2 + k\ (1 \le k \le r)$. Therefore, $\phi_A(k) \le p \cdot l_i(k) + t \le p \cdot (2r^2 - 4r + 2 + k) + t\ (1 \le k \le r)$. □

Let $n = rp + s$ with $0 \le s \le p - 1$ and $r \ge 0$. Note that there is no primitive non-powerful sign pattern matrix satisfying $n \le p$. Thus we exclude the trivial cases $r = 0$ and $r = 1,\ s = 0$.

Theorem 4.7. *Let $n = rp$ and $1 \le i \le r$ $(r \ge 2)$. Then*

$$\phi(n,\ p,\ k) \le p \cdot (2r^2 - 4r + 2 + i), \quad \text{where} \quad p(i-1) + 1 \le k \le p \cdot i.$$

Proof. Let $A = (n_1, A_1, n_2, A_2, \ldots, n_p, A_p, n_1) \in INS_{n,\ p}$ $(p > 1)$.

Case 1. $n_1 = n_2 = \cdots = n_p = r$. Let $l = 2r^2 - 4r + 2 + i$ $(1 \le i \le r)$.

Note that

$$A^{pl} = \begin{pmatrix} [A_1(p)]^l & 0 & \cdots & 0 \\ 0 & [A_2(p)]^l & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [A_p(p)]^l \end{pmatrix}$$

and each $A_j(p)(1 \le j \le p)$ is primitive non-powerful of order r. It follows from Lemma 2.8 that there are i rows $\{u_1^{(j)}, \ldots, u_i^{(j)}\}$ in each $[A_j(p)]^l(1 \le j \le p)$ such that each of whose entries is #.

Let $X = \cup_{j=1}^{p}\{u_1^{(j)}, \ldots, u_i^{(j)}\}$. Then $|X| = p \cdot i$. Take $Y \subseteq X$ with $|Y| = k$, where $p(i-1)+1 \le k \le p \cdot i$. Hence $\beta_A^{pl}(Y) = kr$.

By Theorem 4.4, we have $\beta_A^{pl}(Y) = \beta_A(k)$. It follows that

$$\phi_A(k) \le p \cdot l = p \cdot (2r^2 - 4r + 2 + i), \quad \text{where } p(i-1) + 1 \le k \le p \cdot i.$$

Case 2. There is some integer t such that $n_t \le r - 1$.

Let $l_j = l(A_j(p))(1 \le j \le p)$. By Lemma 2.7, $l_t \le 2(r-1)^2 - 3(r-1) + 2$.

It follows from Lemma 3.5 that $l_j \le l_t + 1 \le 2r^2 - 7r + 8(1 \le j \le p)$.

Since $r \ge 2$, then $2r^2 - 7r + 8 \le 2r^2 - 4r + 2$. By Lemma 4.5,

$$\phi_A(k) \le p \cdot (\max_{1 \le j \le p}\{l_j\}) \le p \cdot (2r^2 - 7r + 8) \le p \cdot (2r^2 - 4r + 2 + i).$$

Therefore, combining Cases 1 and 2, for $n = rp$ and $1 \le i \le r$ $(r \ge 2)$,

$$\phi(n,\ p,\ k) \le p \cdot (2r^2 - 4r + 2 + i), \quad \text{where} \quad p(i-1) + 1 \le k \le p \cdot i. \quad \square$$

Theorem 4.8. *Let $n = rp + s$ with $1 \le s \le p - 1$ and $r \ge 1$. Then*

$$\phi(n,\ p,\ k) \le \begin{cases} p \cdot (2r^2 - 4r + 2 + k) + \max\{1,\ s-1\} & (1 \le k \le r), \\ p \cdot (2r^2 - 3r + 3) & (r + 1 \le k \le n). \end{cases}$$

Proof. Let $A = (n_1, A_1, n_2, A_2, \ldots, n_p, A_p, n_1) \in INS_{n,\ p}$ $(p > 1)$. Let $l_i = l(A_i(p))$ be the base of $A_i(p)(1 \le i \le p)$.

Case 1. $\min_{1 \le i \le p}\{n_i\} < r$. Suppose $n_t = \min_{1 \le i \le p}\{n_i\} \le r - 1$. Then $r \ge 2$.

By Lemma 2.7, we have $l_t \le 2(r-1)^2 - 3(r-1) + 2 = 2r^2 - 7r + 7$.

It follows from Lemma 3.5 that $l_i \le l_t + 1 \le 2r^2 - 7r + 8(1 \le i \le p)$.

Since $r \ge 2$, then $2r^2 - 7r + 8 \le 2r^2 - 4r + 2$. By Lemma 4.5,

$$\begin{aligned} \phi_A(k) &\le p \cdot (\max_{1 \le i \le p}\{l_i\}) \le p \cdot (2r^2 - 7r + 8) \\ &\le \begin{cases} p \cdot (2r^2 - 4r + 2 + k) + \max\{1,\ s-1\} & (1 \le k \le r), \\ p \cdot (2r^2 - 3r + 3) & (r + 1 \le k \le n). \end{cases} \end{aligned}$$

Case 2. $\min_{1 \le i \le p}\{n_i\} = r$.

Subcase 2.1. $s = 1$. Without loss of generality, we assume $n_1 = r + 1$ and $n_2 = n_3 = \cdots = n_p = r$. Therefore, by Lemma 2.7, $l_i \le 2r^2 - 3r + 2(2 \le i \le p)$. And it follows from Lemma 3.5 that $l_1 \le l_i + 1 \le 2r^2 - 3r + 3$.

Since $n_p = r$ and $n_1 = \max_{1\le i\le p}\{n_i\} = r+1,\ 1 \equiv 1-p \pmod p$, by Lemma 4.6, then for every $1 \le k \le r,\ \phi_A(k) \le p\cdot(2r^2-4r+2+k)+1$.

For every $r+1 \le k \le n$, it follows from Lemma 4.5 that

$$\phi_A(k) \le p\cdot(\max_{1\le i\le p}\{l_i\}) \le p\cdot(2r^2-3r+3).$$

Subcase 2.2. $2 \le s \le p-1$. Without loss of generality, we assume $n_1 = \max_{1\le i\le p}\{n_i\} > n_p$. It is obvious that there exists an integer t with $p-s+2 \le t \le p$, such that $n_t = r$. (Otherwise, $n_{p-s+2},\ldots,n_{p-1},n_p \ge r+1$, then $n_1 \ge n_p+1 \ge r+2$. It follows that $n = \sum_{i=1}^{p} n_i \ge rp+s+1$, contracting that $n = rp+s$.)

Hence $l_t \le 2r^2-3r+2$ by Lemma 2.7. Thus $l_i \le l_t+1 \le 2r^2-3r+3 (1\le i\le p)$ by Lemma 3.5. Since $n_t = r$ and $n_1 = \max_{1\le i\le p}\{n_i\},\ 1-t+p \equiv 1-t \pmod p$, by Lemma 4.6, then for every $1\le k\le r$,

$$\phi_A(k) \le p\cdot(2r^2-4r+2+k)+1-t+p \le p\cdot(2r^2-4r+2+k)+(s-1).$$

For every $r+1 \le k \le n$, by Lemma 4.5,

$$\phi_A(k) \le p\cdot(\max_{1\le i\le p}\{l_i\}) \le p\cdot(2r^2-3r+3).$$

In conclusion, combining Cases 1 and 2, for $n = rp+s$ with $1 \le s \le p-1$,

$$\phi(n,\ p,\ k) \le \begin{cases} p\cdot(2r^2-4r+2+k)+\max\{1,\ s-1\} & (1\le k\le r),\\ p\cdot(2r^2-3r+3) & (r+1\le k\le n). \end{cases} \qquad \square$$

Remark 5. Since $\phi(n,\ p) = \phi(n,\ p,\ n)$, it is not difficult to check that the result of Theorem 3.9 can be obtained by Theorems 4.7 and 4.8.

5. The index and the generalized index of maximum ambiguous density for irreducible non-powerful zero-symmetric sign pattern matrices

A square generalized sign pattern matrix A is called zero-pattern-symmetric (abbreviated zero-symmetric, or simply ZS) if $|A|$ is symmetric (see [1]).

Let $SIS_{n,\ p}$ denote the set of $n\times n$ irreducible non-powerful ZS (generalized) sign pattern matrices with period p. Note that the period $p = 1$ or 2.

Combining the results in [1,6], we obtain the following lemma.

Lemma 5.1. *Let $\widetilde{l}(k) = \max\{l_A(k) \mid A \in SIS_{n,\ 1},\ 1\le k\le n\}$. Then*

$$2n-1 \le \widetilde{l}(k) \le 2n.$$

By Lemmas 3.2, 4.2 and 5.1, it is easy to prove that

Corollary 5.2. *Let $A \in SIS_{n,\ 1}$. Then for $1\le k\le n$,*

$$\beta_A(k) = nk \quad \text{and} \quad \phi_A(k) \le 2n.$$

For $A \in SIS_{n,\ 2}$, the result below is a direct consequence of Theorem 4.4.

Theorem 5.3. *Suppose $A = (n_1, A_1, n_2, A_2, n_1) \in SIS_{n,\ 2}$ and $n_1 \ge n_2$. Then*

$$\beta_A(k) = \begin{cases} kn_1 & (1\le k\le n_1),\\ n_1^2+(k-n_1)\cdot n_2 & (1+n_1\le k\le n). \end{cases}$$

In particular, $\beta_A(n) = n_1^2+n_2^2 = \beta(A)$.

Theorem 5.4. *Let* $A = (n_1, A_1, n_2, A_2, n_1) \in SIS_{n,\,2}$ *and* $m = \min\{n_1,\ n_2\}$. *Then*

(1) If $n_1 = n_2$, *then* $\phi_A(k) \le 4m\ (1 \le k \le n)$;

(2) If $n_1 \ne n_2$, *then* $\phi_A(k) \le \begin{cases} 4m+1 & (1 \le k \le m), \\ 4m+2 & (m+1 \le k \le n). \end{cases}$

Proof. Since $A = (n_1, A_1, n_2, A_2, n_1) \in SIS_{n,\,2}$, $A_i(2)$ is an $n_i \times n_i$ primitive non-powerful ZS generalized sign pattern matrix. Let $l_i = l(A_i(2))(1 \le i \le 2)$.

(1) If $n_1 = n_2 = m$, therefore, by Lemma 5.1, we have $l_1 = l_2 \le 2m$.

For every $1 \le k \le n$, by Lemma 4.5, we have

$$\phi_A(k) \le 2 \cdot (\max\{l_1,\ l_2\}) \le 2 \cdot (2m) = 4m.$$

(2) If $n_1 \ne n_2$, without loss of generality, suppose $n_2 = m = \min\{n_1,\ n_2\}$.

Let $l_i(k) = l_{A_i(2)}(k)$ be the kth local base of $A_i(2)(1 \le i \le 2)$. Therefore, by Lemma 5.1, $l_2(k) \le 2m$. It follows from Lemma 3.5 that $l_1 \le l_2 + 1 \le 2m + 1$.

Since $n_2 = m$ and $n_1 = \min\{n_1,\ n_2\}$, $1 \equiv -1 \pmod 2$, $l_2(k) \le 2m$, by Lemma 4.6, for every $1 \le k \le m$, we have $\phi_A(k) \le 2 \cdot (2m) + 1 = 4m + 1$.

For every $m + 1 \le k \le n$, it follows from Lemma 4.5 that

$\phi_A(k) \le 2 \cdot (\max\{l_1,\ l_2\}) \le 2 \cdot (2m + 1) = 4m + 2$. $\square$

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